Counting points on Elliptic Curve defined over GF(2^n) and its software implementation

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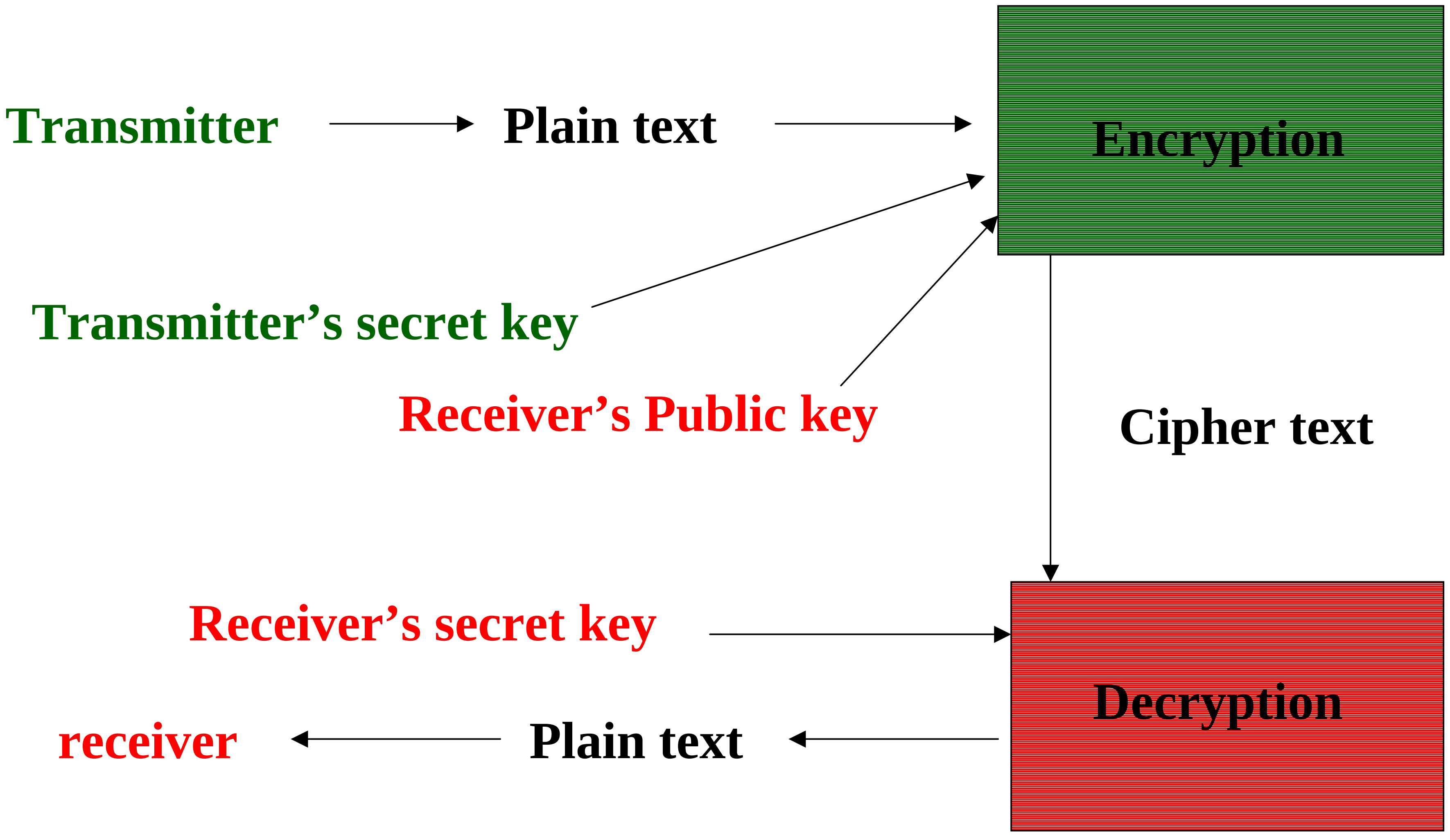
Today’s Contents

- PKC and One way function
- DLP on Elliptic Curve and Number
- 2-adic Numbers and its Valuation Ring R
- Bit-Slice technique
- Addition,Multiplication and Inversion in Z2 and R
- Running times in Z2 and R
- Sato-Skjernna-Taguchi (SST) Algorithm
- Hasse’Theorem ,Isogeny, Frobenius morphism
- Lifting the j-invariant and 2-torsion,compute trace
- Running Times over GF(2^7)

Public key cryptosystem



One Way function

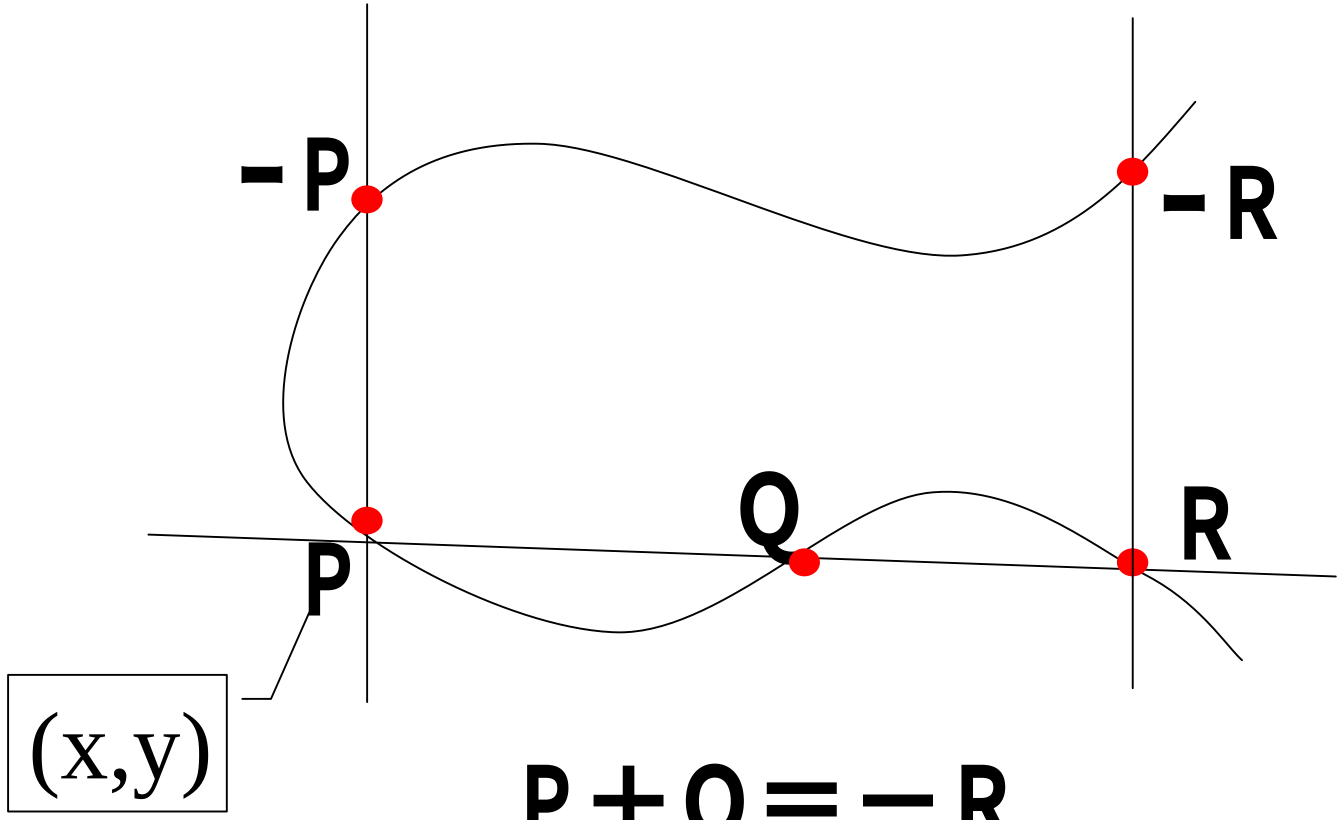
$$x_{public} = f(x_{secret})$$

$$\cancel{\Downarrow}$$

$$f^{-1}(x_{public}) = x_{secret}$$

- Assume the function " f " is public. The inverse function of " f " can't be solved from public information. ➔one way function
- Secret information can be gained by special way using each user's secret key.

Elliptic Curve over Real Number



$$P + Q = -R$$

$$E: y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

Discrete logarithm problem on Elliptic Curve over Finite Field

$$k \times P = Q \quad \bmod \# E$$

k:Real Number (not zero)
P,Q:point on Elliptic curve over finite field
#E:The number of points on Elliptic curve

- Let P be public.Then,heavy computation is needed to get k from given Q . ➔DLP on EC
- The more the number of points are,the more difficult to solve DLP on EC is.
- Counting points on random Elliptic Curves is needed and must be fast applying CRYPTOGRAPHY.

Elliptic Curves over GF(2^n)

- Why GF(2^n)?
  - It’s easy to implement on computer because binary representation
- The form is following,

$$E: y^2 + xy = x^3 + a_6 \qquad a_6 \in GF(2^n)$$

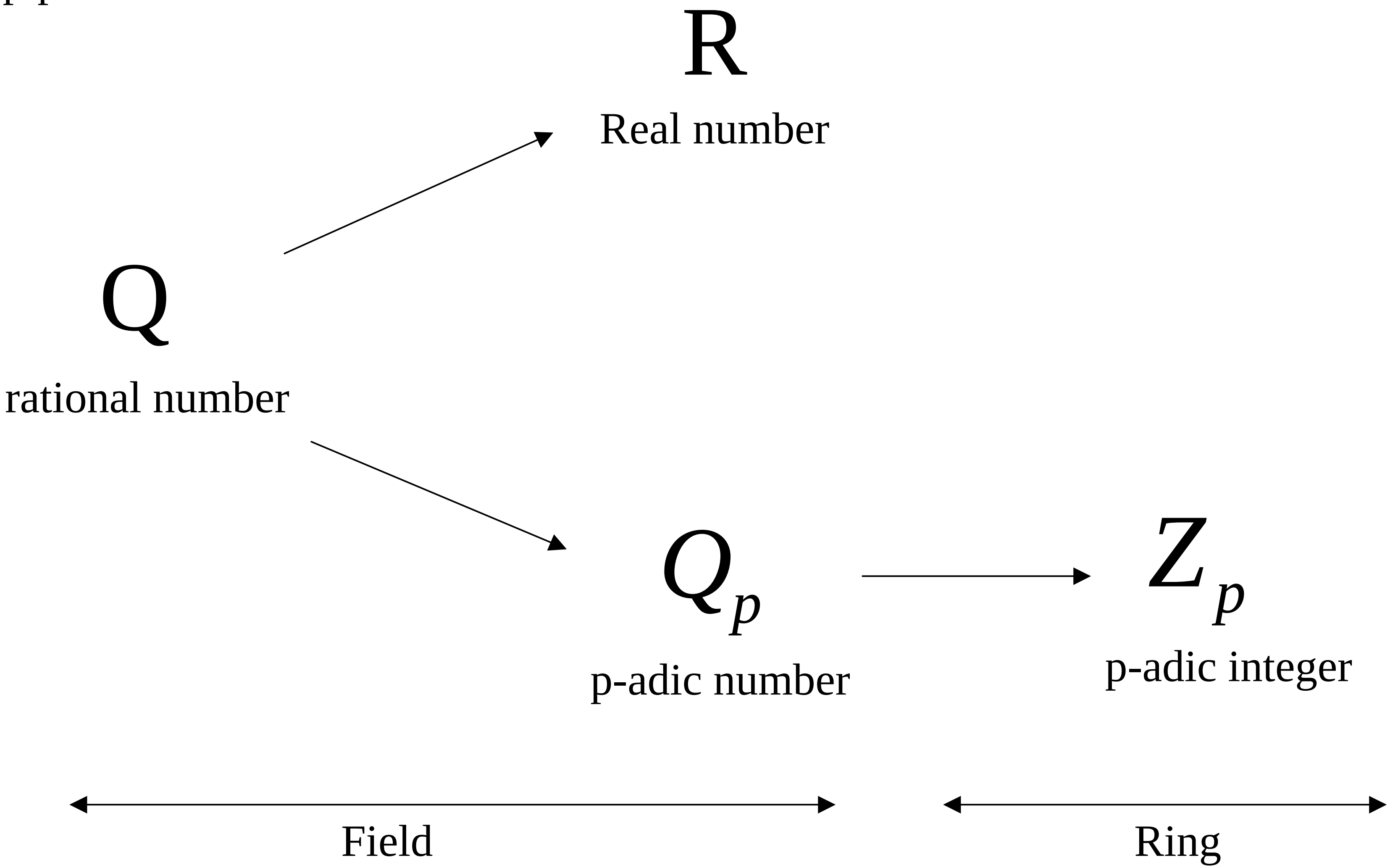
- j-invariant is defined by :

$$j(E) = 1/a_6 \qquad j(E) \in GF(2^n)$$

This is the characteristic value for the each curve.

Numbers

p:prime



The Ring Z_2 — 2-adic integer

- Definition (2-adic integer)
  - Let π_n be the projection $Z/2^{n+1}Z \to Z/2^n Z$
  - A sequence $x = (x_1 \ldots x_n \ldots)$,with $x_n \in Z/2^n Z$ and such that $\pi_n(x_{n+1}) = x_n$ for $n \geq 1$.
  - The ring of 2-adic integers is denoted by Z_2 .
- More precisely, $x \in Z_2$,

$$x = x_0 2^0 + x_1 2^1 + x_2 2^2 + \cdots + x_n 2^n + \cdots \qquad x_i = \{0,1\}$$

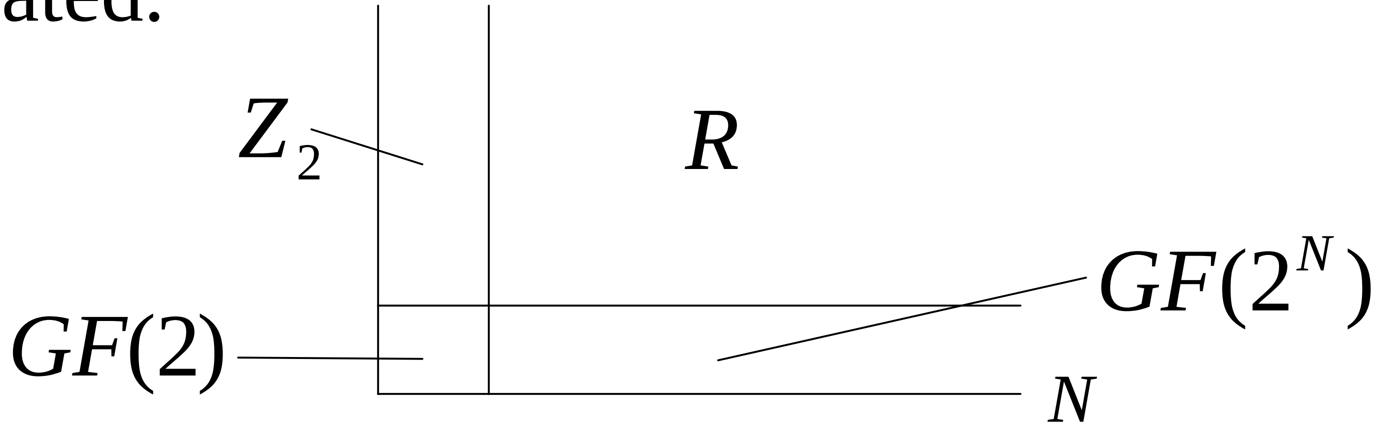
- The index "n" is larger, the absolute value is smaller.
  - For example, $2^0 + 2^1 > 2^0 + 2^2$ that is, 3>5.

Valuation Ring R

- Let f(t) be a monic polynomial Z_2[t] of degree N such that the polynomial $\pi(f)$ obtained by projecting the coefficients is irreducible in GF(2^N).
- Valuation Ring R (Definition):

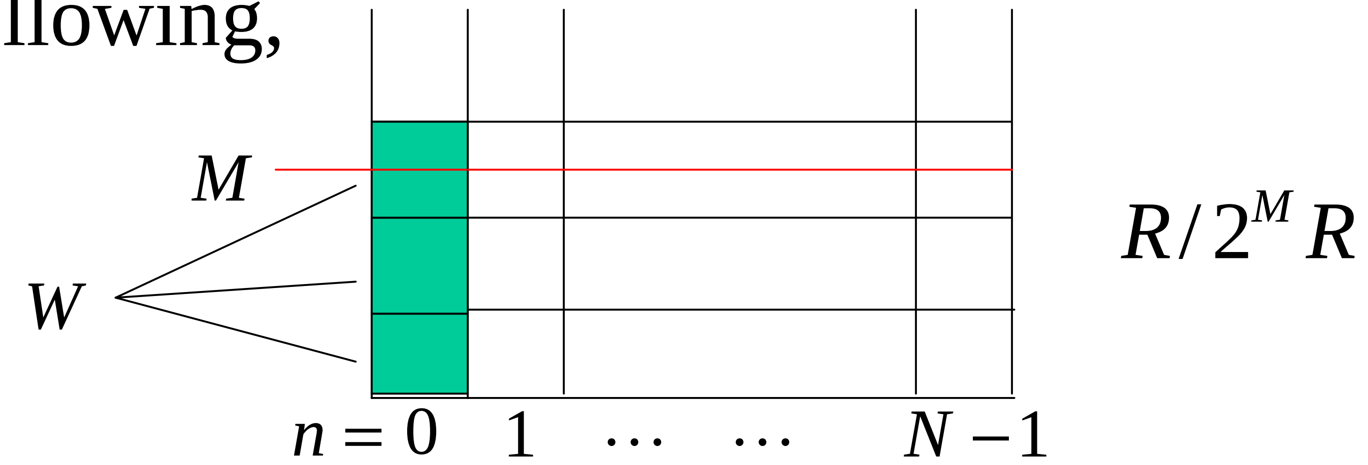
$$Z_2[t] \bmod f(t)$$

- As following diagram, GF(2),Z_2 ,R and GF(2^N) are related.



Valuation Ring R on computer

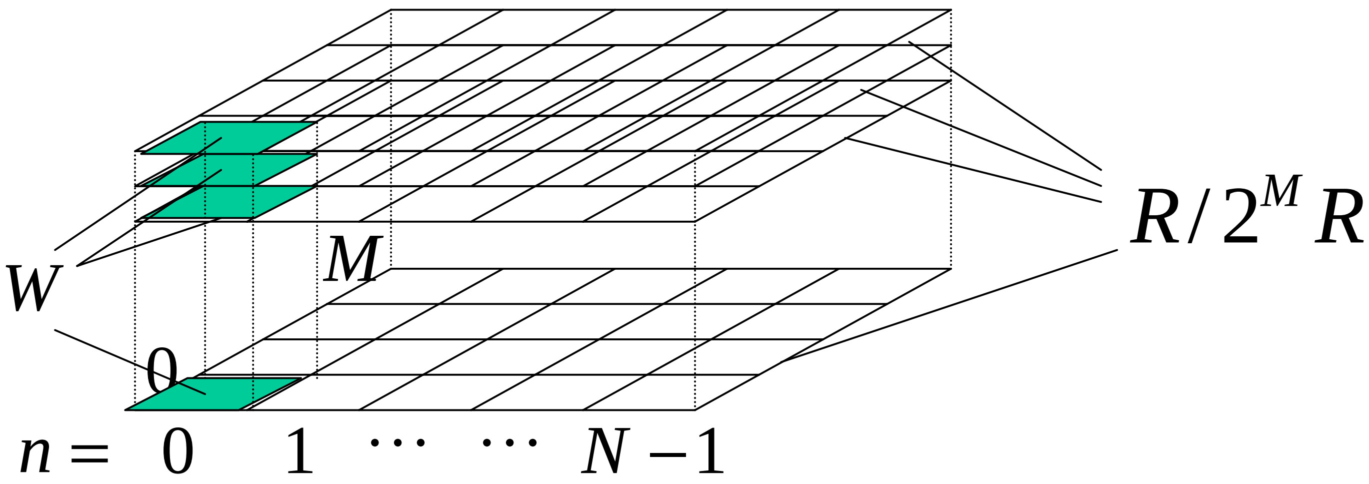
- The index "n" of Z_2 (called "precision") must be finite to implement on computer.
- Normally,Valuation Ring R can be implemented as following,



Where M is a required precision and W is WORD SIZE of CPU.

- We must treat Z_2 as Multi precision integer .

Bit Slice Technique



- W data are implemented simultaneously and this can be reduced redundancy in last one word of precision.
- Memory requirement is $N \times M$ words.
- An algorithm has any condition branches cannot be applied.

Bit Slice Addition in Z_2

- Let $x, y \in Z_2$ be 2-adic integers,then 2-elements addition is represented as following.

$$x = x_0 2^0 + x_1 2^1 + x_2 2^2 + \cdots + x_{n-1} 2^{n-1} + \cdots \qquad x_i = \{0,1\}$$

$$y = y_0 2^0 + y_1 2^1 + y_2 2^2 + \cdots + y_{n-1} 2^{n-1} + \cdots \qquad y_i = \{0,1\}$$

$$\begin{aligned} x + y = {} & (x_0 \oplus y_0) 2^0 + ((x_0 \,\&\, y_0) \oplus x_1 \oplus y_1) 2^1 + \\ & \cdots + [\{(x_{n-2} \,\&\, y_{n-2}) \oplus (x_{n-2} \,\&\, (x_{n-3} \,\&\, y_{n-3})) \\ & \oplus (y_{n-2} \,\&\, (x_{n-3} \,\&\, y_{n-3}))\} \oplus x_{n-1} \oplus y_{n-1}] 2^{n-1} + \cdots \end{aligned}$$

- (4n-5) times XOR operation and (5n-9) times AND operation are needed for 2-adic integers addition which precision is n (n$>$2).

Bit Slice Subtraction in Z_2

- Let $x, y \in Z_2$ be 2-adic integers,then 2-elements subtraction is represented as following.

$$x = x_0 2^0 + x_1 2^1 + x_2 2^2 + \cdots + x_{n-1} 2^{n-1} + \cdots \quad x_i = \{0,1\}$$
$$y = y_0 2^0 + y_1 2^1 + y_2 2^2 + \cdots + y_{n-1} 2^{n-1} + \cdots \quad y_i = \{0,1\}$$
$$x - y = x + 1 + \{(1 \oplus y_0)2^0 + (1 \oplus y_1)2^1 + \cdots + (1 \oplus y_{n-1})2^{n-1} + \cdots\}$$

where –y is represented as two’s complement of y and x-y can be calculated as additions in Z_2 .

- (9n-10) times XOR operation and (10n-18) times AND operation are needed for 2-adic integers subtraction which precision is n (n>2).

Bit Slice ADD and SUB in R

- Let X,Y be 2-adic Valuation Ring, then 2 elements addition and subtraction are represented as following.

$$X = X_0 + X_1\psi + X_2\psi^2 + \cdots + X_{n-1}\psi^{n-1} \quad X_i \in Z_2$$

$$Y = Y_0 + Y_1\psi + Y_2\psi^2 + \cdots + Y_{n-1}\psi^{n-1} \quad Y_i \in Z_2$$

$$X + Y = (X_0 + Y_0) + (X_1 + Y_1)\psi + \cdots + (X_{n-1} + Y_{n-1})\psi^{n-1}$$

- N times addition (subtraction) in Z_2 can realize ADD (SUB) in R.

Bit Slice Multiplication in Z_2

- Let $x, y \in Z_2$ be 2-adic integers,then 2-elements multiplication is represented as following.

$$x = x_0 2^0 + x_1 2^1 + x_2 2^2 + \cdots + x_n 2^n + \cdots \qquad x_i = \{0,1\}$$
$$y = y_0 2^0 + y_1 2^1 + y_2 2^2 + \cdots + y_n 2^n + \cdots \qquad y_i = \{0,1\}$$

$$\begin{aligned} x \times y = {} & x_0 \,\&\, (y_0 2^0 + \cdots + y_{n-1} 2^{n-1}) \\ & + x_1 2^1 \,\&\, (y_0 2^0 + \cdots + y_{n-1} 2^{n-1}) \\ & \cdots + x_n 2^{n-1} \,\&\, y_0 2^0 \end{aligned}$$

- **Σ**(4k-5) times XOR operation and **Σ**(5k-9)+n(n+1)/2 times AND operation are needed for 2-adic integers multiplication which precision is n (n>2).

Efficient Multiplication in R

- Karatsuba Method (1962)

Let $X, Y \in R$ then,

$$X \times Y = (X_A \psi^k + X_B)(Y_A \psi^k + Y_B)$$

$$= X_A Y_A \psi^{2k} + X_B Y_B + (X_A Y_B + X_B Y_A)\psi^k$$

$$= X_A Y_A \psi^{2k} + X_B Y_B$$

$$-\{X_A Y_A + X_B Y_B - (X_A + X_B)(Y_A + Y_B)\}\psi^k$$

- If two elements have 2k-length,its multiplication can be realized 3-times k-length MUL and some ADD.
- The iteration control structure can be used.

Efficient Multiplication in Z_2

- Modified Karatsuba Method

Let $x, y \in Z_2$ and their precision be M.Let

$$Kara(x, y) = x_A y_A 2^{2k} + x_B y_B$$
$$-\{x_A y_A + x_B y_B - (x_A + x_B)(y_A + y_B)\}2^k$$
$$Mkara(x, y) = x_B y_B + (x_A y_B + x_B y_A)2^k \ (k > M/2)$$

Then,

$$x \times y = (x_1 2^k + x_2)(y_1 2^k + y_2)$$
$$= Kara(x_2, y_2)$$
$$+ (Mkara(x_1, y_2) + Mkara(x_2 + y_1))2^k$$

Newton iteration and inversion

- Quadratic convergence of Newton iteration

Let $x \in R$ and $f(x) \in R[t]$.Let k be such that $2^k \| f'(x)$ and assume $2^{n+k} \mid f(x)$ for some n>k.Let

$$\Delta = \frac{2^{-k} f(x)}{2^{-k} f'(x)} \qquad y = x - \Delta$$

Then $y \equiv x \bmod 2^n$, $2^{2n} \mid f(y)$ and $2^k \| f'(y)$

- Inverse of an invertible $a \in R$ can be obtained by Newton iteration whose f(x)=1/x-a.That is,

$$x \leftarrow x - f(x) / f'(x) = x + x(1 + ax)$$

Running Times

CPU	PentiumⅢ 1GHz (seagull)
Second cache	256KB
Main memory	512MB
Programming lang.	C++
Compiler	gcc version 2.95.3
Compile option	-O3

Running Times in Z_2 (pre=95)

Operation	BITSLICE (*32) [μs]	BITSLICE (*1) [μs]	NORMAL (*1) [μs]
ADD	**5.4**	**0.169**	**0.709**
SUB	**17.8**	**0.556**	**0.734**
MUL	**130.1**	**4.066**	**5.17**
INV	7500	234.8	No data

Running Times in R (deg=163)

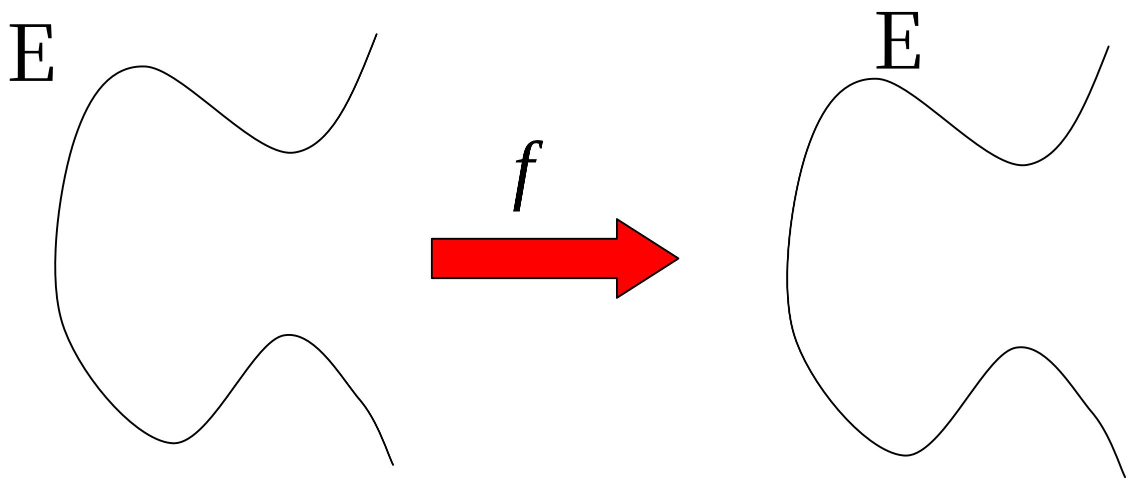
Operation	BITSLICE (*32) [μs]	BITSLICE (*1) [μs]	NORMAL (*1) [μs]
ADD	3099	96.84	231
SUB	3314	103.7	168
MUL	5224000	163250	144000
Karatsuba MUL	under construction	under construction	4100

How to count points -SST Algorithm

- Sato-Skjernna-Taguchi(SST) Alghorithm
  - Time complexity: O(N^3.5)
  - Memory complexity:O(N^2.5)
- The process is following,
  - Lift j-invariant from GF(2^n) to R
  - Determine the Kernel of 2-th Verschiebung V
  - Let τ The local parameter at the point at infinity. Then find the value c^2 from expansion V(τ)=c τ +O(τ ^2)
  - Find an integer t satisfying t^2=Norm(c^2)
  - Finally we get #E=1+2^n-t !

Isogeny (Endomorphism)

- (definition) A rational map which is furthermore a group homomorphism



$$\varphi : P(x,y) \mapsto (x^{2^N}, y^{2^N})$$

$$[m] : P \mapsto mP$$

Hasse’s Theorem

- Hasse’s Theorem(1934)

$$\# E = 2^{N} + 1 - t$$

$$-2\sqrt{2^{N}} < t < 2\sqrt{2^{N}}$$

- What is “ t ” ??

$$\varphi \circ \varphi + [t] \circ \varphi + [2^{N}] = [0]$$

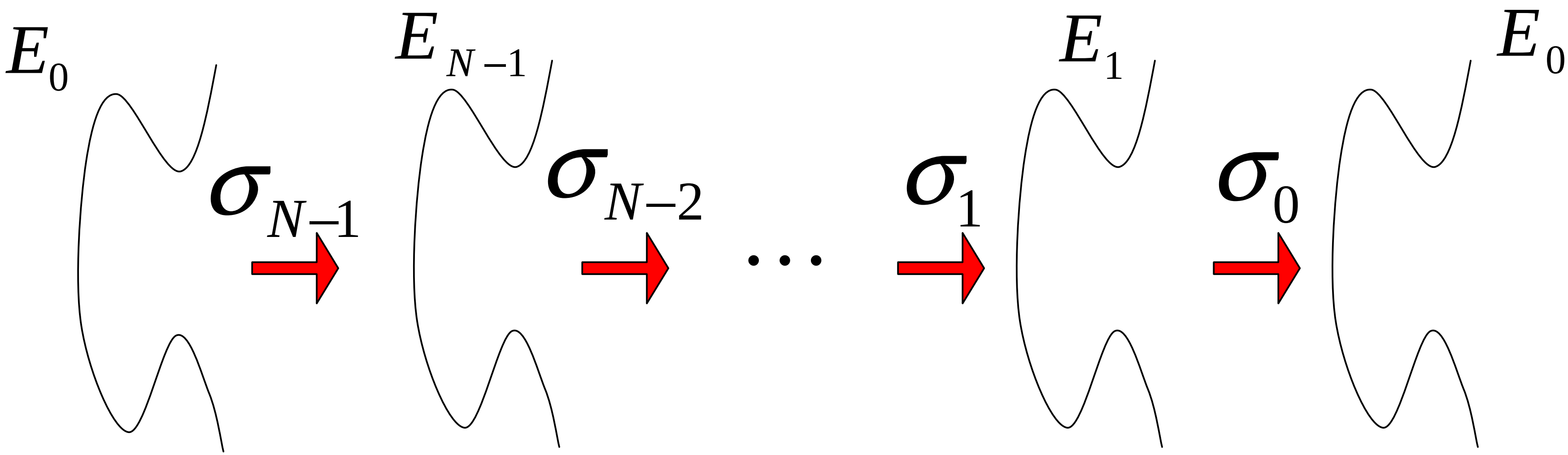
$$\varphi : Frobenius \quad endomorphism$$

$$[m] : m - maps$$

The little Frobenius

- 2-th power Frobenius (the little Frobenius) σ is defined as follows.

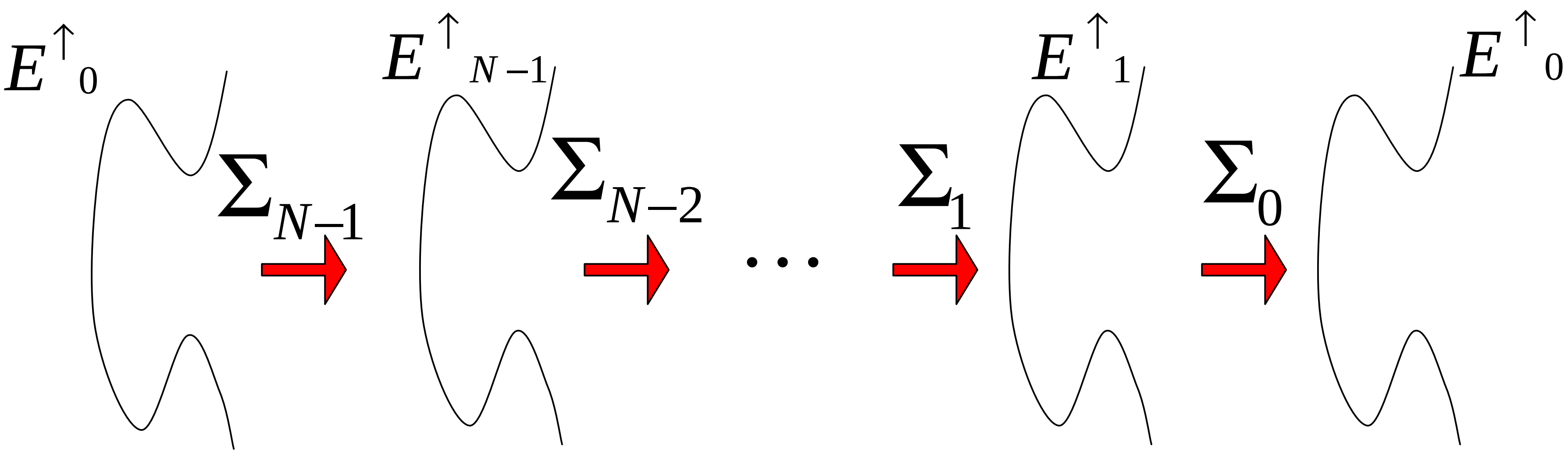
$$\sigma : P(x, y) \mapsto (x^2, y^2)$$



$$\varphi = \sigma_0 \sigma_1 \cdots \sigma_{N-1}$$

Canonical lift of E

- The canonical lift of an ordinary elliptic curve E is unique elliptic curve E↑ defined over R,which satisfies:
  - The reduction of E↑ is E.
  - End(E)=End(E↑)



Modular polynomial and lifting j-inv

- 2-th modular polynomial is defined as:

$$\Phi_2(X,Y)=X^3+Y^3-X^2Y^2+2^4\cdot 3\cdot 31(X^2Y+XY^2)$$
$$-2^4\cdot 3^4\cdot 5^3(X^2+Y^2)+3^4\cdot 5^3\cdot 4027XY+2^8\cdot 3^7\cdot 5^6(X+Y)-2^{12}\cdot 3^9\cdot 5^9$$

- If two Elliptic curves E and E’ are related via a cyclic isogeny of degree N,

$$\Phi_2(j(E),j(E'))=0$$

- (Lubin-Serre-Tate)If J is the j-invariant of the canonical lift of E,then there is a unique J in R such that

$$\Phi_2(J,\Sigma(J))=0 \quad \textit{and} \qquad J\equiv j \quad \mathrm{mod}\, 2$$

Lift the j-invariant

- The process of lifting the j-invariant is iterative Newton's method.

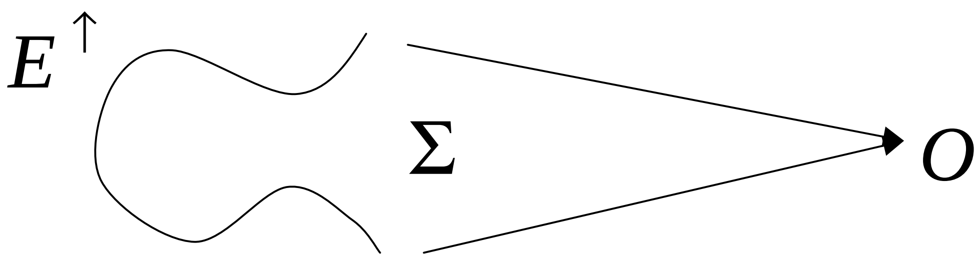
$$for(\text{int } i = 0; i < PRECISION\ ; i++)\{$$

$$x = \Sigma(J) \quad \bmod 2^{i+1};$$

$$J = J - \frac{\Phi_2(x, J)}{\partial_x \Phi_2(x, J)} \quad \bmod 2^{i+1};$$

$$\}$$

Lifting the Kernel of Σ



- The Kernel of Σ is 2-torsion point of E↑ i.e. infinity and another non-trivial point.
- x coordinates of non-trivial point can be computed by j-invariant of E↑ and E'↑.

$$\frac{x}{2} = -\frac{(J'^2 + 195120J' + 4095J + 660960000)/2^{12}}{(J'^2 + J'(563760 - 512J) + 372735J + 8981280000)/2^{9}}$$

Computing the trace

- Let **τ** be the local parameter of E↑ around infinity .then, **Σ** (**τ**) can be expanded as:

$$\Sigma(\tau) = c\tau + \mathrm{O}(\tau^2)$$

- c is computed by:

$$c^2 = \frac{J - (504 + 12096\, z)t}{J + 240\, t}$$

where, $z = x/2$, $t = (12z^2 + z)(J - 1728) - 36$

- Trace t is square root of

$$t^2 \equiv \prod_{0 \le i \le N} c^2 \quad \mod 2^{PRECISION - 1}$$

Applicable??

- INPUT：j-invaritant ∈GF(2^N)

  OUTPUT：trace ∈ Z_2

  ↓

  It's easy to translate bit-slice representation to normal one.

- Algorithm has NO Condition branche.

  ↓

- .We can apply bit-slice technique!

Running Times over GF(2^7)

Irreducible polynomial	**t^7+t+1**
curve	**Y^3+xy=x^3+1/j**
j-invariant	**t^5+t+1**
trace precise (work precise)	**2^6 (2^17)**
trace	**-3**
#E	2^7+1-3=126
Time	1.454/32=0.0454[s]

Future works

- Counting points on EC over Large extension fields
- Implementing Karatsuba method
- Implementing faster algorithm of Lifting the j-invariant and Norm computing

…Thank you for time !

Addition in Z_2

- Let $x, y \in Z_2$ be 2-adic integers,then 2-elements addition is represented as following.

$$x = x_0 b^0 + x_1 b^1 + x_2 b^2 + \cdots + x_{n-1} b^{n-1} + \cdots \quad 0 \le x_i \le b$$

$$y = y_0 b^0 + y_1 b^1 + y_2 b^2 + \cdots + y_{n-1} b^{n-1} + \cdots \quad 0 \le y_i \le b$$

$$x + y =$$

- (4n-5) times XOR operation and (5n-9) times AND operation are needed for 2-adic integers addition which precision is n (n$>$2).