Scalar-field approach of IE-MEI Method

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Outline

- Background
- Motivation of SIE-MEI
- SIE-MEI method
- Numerical Example
- Result
- Remarks & Future work
Background

Wave Scattering Problem

Time domain
- DE approach
  - FD/FE method

Frequency domain
- IE approach
  - MEI method
  - MEI method
  - IE-MEI method
    - SIE-MEI method

Sparse Matrix
FD/FE Method : DE Approach

Large Sparse Matrix

\[
[A] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 & 0 & \cdots & a_{1(n-1)} & a_{1n} \\
a_{21} & a_{22} & a_{23} & a_{24} & 0 & \cdots & 0 & a_{2n} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & 0 & \cdots & a_{n(n-2)} & a_{n(n-1)} & a_{nn}
\end{bmatrix}
\]
BEM Method : IE Approach

\[
[A] = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
    a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{bmatrix}
\]
**MEI Method : DE Approach**

(K.K.Mei et al., 1994)

Local linear equation of MEI node,

\[ \sum_{i=1}^{4} C_i \phi_i = 0 \]

MEI coefficients:
1. location dependent,
2. geometry specific, and
3. invariant to field excitation.

MEI postulates

scattered field

FD node

MEI node

Truncation
**IE-MEI Method : IE Approach**

(J.M.Rius et al., 1996 & M.Hirose et al., 1999)

Surface IE is derived from Reciprocity relation
On surface MEI postulates
Sparse matrix with same number of unknowns as BEM

Savings in Computational time, and Memory needs

Suitable for arbitrary 2D boundaries, but not efficient for 3D boundaries
Motivation of SIE-MEI Method

IE-MEI method has an improvement to store and solve the matrix efficiently

Not suitable for arbitrary shape 3D problem

Choice of suitable metrons and mesh generation for 3D, yet not established

To approach this problem introduce scalar-field IE-MEI for the 3D boundaries
**SIE-MEI Method**

- Scalar-field problem
- Scalar-field Integral Eqn.
- MEI method
- Scalar Helmholtz Equations
- Hirose’s approach
- Scalar Reciprocity Relation
- Least Square solution
- System of linear equations
- Boundary condition
- Derive the desired solution
- Sparse Matrix of Unknown Coefficients
Scalar-field problem:

\[ L\{\phi(\mathbf{r})\} = -g(\mathbf{r}) \]

Scalar Helmholtz Equation

\[
\begin{align*}
\nabla^2 \phi_1(\mathbf{r}) + k^2 \phi_1(\mathbf{r}) &= -g_1(\mathbf{r}) \\
\nabla^2 \phi_2(\mathbf{r}) + k^2 \phi_2(\mathbf{r}) &= -g_2(\mathbf{r})
\end{align*}
\]

Scalar Reciprocity relation

\[
\int_V (\phi_2(\mathbf{r})g_1(\mathbf{r}) - \phi_1(\mathbf{r})g_2(\mathbf{r}))dV = \int_{\partial V} \left( \phi_1(\mathbf{r}) \frac{\partial \phi_2(\mathbf{r})}{\partial n} - \phi_2(\mathbf{r}) \frac{\partial \phi_1(\mathbf{r})}{\partial n} \right) dS
\]
Integral Equation formulation:

**scattered field**

combination of equivalent surface source $\rho_2(\mathbf{r})$ and monopole of dipole moment $\mu_2(\mathbf{r})$

\[
\oint_{\partial V^+} \left( \phi_1(\mathbf{r}) \frac{\partial \phi_2(\mathbf{r})}{\partial n} - \phi_2(\mathbf{r}) \frac{\partial \phi_1(\mathbf{r})}{\partial n} \right) dS = - \int_{V^+} \phi_1(\mathbf{r}) \rho_2(\mathbf{r}) dV
\]

$V^+ \rightarrow V$

\[
\oint_{\partial V} (\phi_1(\mathbf{r}) \tilde{\rho}_2(\mathbf{r}) - \frac{\partial \phi_1(\mathbf{r})}{\partial n} \tilde{\mu}_2(\mathbf{r}) \cdot \hat{n}) dS = 0
\]

equivalent local sources near the scatterer
Localization & Discretisation:

Localyze the IE which satisfies MEI postulates

\[ \int_{S_0} (\phi_1(\mathbf{r})\tilde{\rho}_2(\mathbf{r}) - \frac{\partial \phi_1(\mathbf{r})}{\partial n} \tilde{\mu}_2(\mathbf{r}) \cdot \hat{n})dS = 0 \]

By discretization and expanding the local sources around \( \mathbf{r}_n \)

unknown MEI coefficients

\[ \sum_{m=n-M/2}^{n+M/2-1} \left[ \phi_1(\mathbf{r}_m) \tilde{\rho}_{2,n}(\mathbf{r}_m) - \frac{\partial \phi_1(\mathbf{r}_m)}{\partial n} \tilde{\mu}_{2,n}(\mathbf{r}_m) \cdot \hat{n} \right] = 0 \]

depends on scatterer geometry, depends on position, and invariant to incident field
Derivation of MEI coefficients:

\[
\sum_{m=n-M+1}^{n+M-1} \left[ \phi_1(r_m) \tilde{\rho}_{2,n}(r_m) - \frac{\partial \phi_1(r_m)}{\partial n} \tilde{\mu}_{2,n}(r_m) \cdot \hat{n} \right] = 0
\]

\[
\phi_{1,q}(r_m) = \int_s \rho_q(r') G(r_m, r') \, ds'
\]

called Metrons

\[ q = 1, 2, \ldots, N_{max} \]

In matrix form,

\[
[A \ B] \begin{bmatrix} r \\ m \end{bmatrix} = 0
\]

\[ [q \times 2M] \text{ matrix of scattered field} \]

\[ \text{column matrix of } 2M \text{ unknown MEI coefficient} \]

\[
G(r, r') = \frac{e^{-jk|r - r'|}}{4\pi |r - r'|}
\]
Green’s function: \( G(\mathbf{r}, \mathbf{r}') \)

Green’s function \( G(\mathbf{r}, \mathbf{r}') \), is the response at a observer \( \mathbf{r} \) from a unit point source at \( \mathbf{r}' \).

The total field generated by the distributed source can be obtained by the integration of the Green’s function weighted by the source distribution.

**Free space Green’s function for the Scalar Helmholtz equation**

**2D case**

\[
G(\rho, \rho') = \frac{1}{4j} H_0^{(2)}(k|\rho - \rho'|)
\]

**3D case**

\[
G(\mathbf{r}, \mathbf{r'}) = \frac{e^{-jkR}}{4\pi R} = \frac{e^{-jk|\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|}
\]
Solve this Matrix equation according to Least square solution.

Coefficients for particular nodal point $n$, $r_1$, $r_2$, $\cdots$, $r_M$ and $m_1$, $m_2$, $\cdots$, $m_M$.
Repeat the procedure for each nodal point, $n = 1, \ldots, N$ and get the Sparse matrix $R$ and $M$.

**Global Matrix**

$$R = \begin{bmatrix}
    r_{1,1} & r_{1,2} & r_{1,3} & 0 & \cdots & r_{1,n-1} & r_{1,n} \\
    r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} & \cdots & 0 & r_{2,n} \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    r_{n,1} & r_{n,2} & 0 & \cdots & r_{n,n-2} & r_{n,n-1} & r_{n,n}
\end{bmatrix}$$

For $M = 5$

$$M = \begin{bmatrix}
    m_{1,1} & m_{1,2} & m_{1,3} & 0 & \cdots & m_{1,n-1} & m_{1,n} \\
    m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & \cdots & 0 & m_{2,n} \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    m_{n,1} & m_{n,2} & 0 & \cdots & m_{n,n-2} & m_{n,n-1} & m_{n,n}
\end{bmatrix}$$
Scattered field for Soft Surface:

\[ \phi_0^{inc}(\mathbf{r}_s) + \phi_0^{sc}(\mathbf{r}_s) = 0 \quad \mathbf{r}_s \in \partial V \quad \text{Boundary condition} \]

\[
\begin{bmatrix} R \end{bmatrix} \begin{bmatrix} \phi_0^{inc} \end{bmatrix} + \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_0^{sc}}{\partial n} \end{bmatrix} = 0 \quad \text{Boundary condition in Matrix equation}
\]

\[
\frac{\partial \phi_0(\mathbf{r}_n)}{\partial n} = \frac{\partial \phi_0^{inc}(\mathbf{r}_n)}{\partial n} + \frac{\partial \phi_0^{sc}(\mathbf{r}_n)}{\partial n}
\]

\[
= \begin{bmatrix} \frac{\partial \phi_0^{inc}(\mathbf{r}_n)}{\partial n} \end{bmatrix} - \begin{bmatrix} M \end{bmatrix}^{-1} \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} \phi_0^{inc} \end{bmatrix}
\]

\[ \phi_0^{sc}(\mathbf{r}) = \sum_{n=1}^{N} \frac{\partial \phi_0(\mathbf{r}_n)}{\partial n} G(\mathbf{r}, \mathbf{r}_n) \Delta s \quad \text{Derivation for Scattered field} \]
Numerical Example

Incident field,

\[ \phi_o^{inc}(r) = e^{-jka \cos \theta} \]

Scattered field,

\[ \phi_o^{sc}(r) = \int_s \rho_q(r') G(r, r') \, ds' \]

Zonal Harmonics as Metrons,

\[ \rho_q = P_q(\cos \theta), \quad q = 1, 2, \ldots, N_{max} \]
Localization and Discretization:

Measuring function varies only in polar direction, $\theta$.

Local sources expanded into, $M = 3$.

Use rectangular patches for discretization.

Discretized surface area,

$$ds = a^2 \sin \theta \, d\theta \, d\varphi$$

Segment length, $\lambda = \frac{1}{10}$ (wavelength)
Singular value treatment:

When $\mathbf{r}$ and $\mathbf{r}'$ are on the same patch, i.e., $\mathbf{r} \rightarrow \mathbf{r}'$

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jkR}}{4\pi R} = -\frac{jk}{4\pi} h_o^{(2)}(kR), \quad R = |\mathbf{r} - \mathbf{r}'|$$

\[
\int_{\Delta S_s} h_o^{(2)}(kR_s) dS' = a^2 \sin \theta \Delta \theta \Delta \varphi + j\frac{2a \Delta \varphi}{k} \ln \left[ \mathbf{v} + (\mathbf{v}^2 + 1)^{\frac{1}{2}} \right]
\]
\[
+ j\frac{2a \sin \theta \Delta \varphi}{k} \ln \left[ \mathbf{v}^{-1} + (\mathbf{v}^{-2} + 1)^{\frac{1}{2}} \right]
\]

where, $\mathbf{v} = \frac{\sin \theta \Delta \varphi}{\Delta \theta}$
Result

Plot of Normal derivative of the field on the Sphere by using SIE-MEI

Radius = 5 wavelength (λ)
Plot of Near Scattered field

Radius = 5 wavelength (\(\lambda\))
Present Work

Fictitious Surface Source distribution by using CfMoM

Magnitude

Phase[deg]

--> $\lambda$

--> $\lambda$
Remarks & Future Work

SIE-MEI is derived from IE-MEI method; Savings in CPU time and memory requirement

Successfully implemented to uniform shape 3D scalar field (Acoustic) problem

Numerical results has an excellent agreement with the analytical solution

In arbitrary shape 3D problem, SIE-MEI method can be applied without any significant difference